CHAPTER 10

SIMPLE CLIMATE MODELS

10.1 Abstract

Simple, one dimensional climate models are commonly used in the first studies of a planet's climate. While planets in our solar system are nowadays usually modelled by general circulation models, simple climate models, which require minimal observational constraints on the nature of an atmosphere, will be useful in studies of extra-solar planets.

This Chapter has two distinct parts. In Section 10.2 I investigate a two-box model of an atmosphere which uses a variational principle to predict heat transport within the atmosphere. I derive an analytical expression that relates the efficiency with which energy is radiated to space to that with which it is transported within the atmosphere. This expression has a surprisingly simple form in most cases. It can be used to make predictions which can then be compared with observations to test the simple atmospheric model. I do not compare predictions with observations because investigating the validity of the model is not the purpose of this work. This work is an investigation of some properties of the previously proposed model, not a test of the model.

In Section 10.3 I investigate three other simple climate models that have been proposed in the literature and show that their predictions are almost identical in most cases. This demonstrates that testing one of them is almost equivalent to testing all of them and that it will be very difficult for other workers to prove that one model is true and the other two are false.

10.2 Two-Box Model

Lorenz et al. (2001) discussed a two-box model for climate controlled by a variational principle. Variational principles (*e. g.* the principle of least action in mechanics or Fermat's principle of least time in optics) are a well-established part of physics, though their discoveries are often serendipitious (Paltridge, 1975; Yourgrau and Mandelstam, 1960).

The model of Lorenz et al. (2001) is shown in Figure 10.1. Energy balance requires:

$$I_0 - F - E_0 = 0 \tag{10.1}$$

$$I_1 + F - E_1 = 0 \tag{10.2}$$

In this model, I_0 and I_1 are assumed to be known from the planet's orbital properties and albedo. E is assumed to be a known linear function of temperature. F, T_0, and T_1 are unknowns to be solved for.

$$\begin{aligned} E_0 &= A + BT_0 \\ E_1 &= A + BT_1 \end{aligned} \tag{10.3}$$

10.2.1 Variational Principle

The model currently has one fewer equations (10.1 and 10.2) than unknowns (F, T_0, T_1). The hypothesized variational principle provides the final constraint. It is hypothesized that the rate of entropy production $(F/T_1 - F/T_0)$ within the

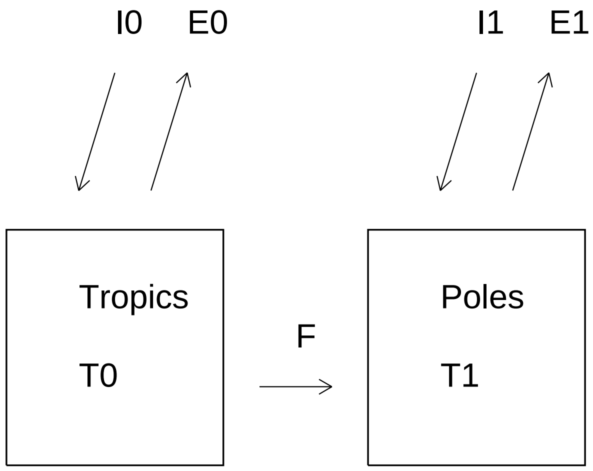

Figure 10.1: Simple two-box climate model. The boxes are of equal surface area, the left (subscript "0") corresponding to regions equatorward of 30° latitude and the right (subscript "1") to regions poleward of 30° latitude. I is absorbed solar flux, E is outgoing thermal flux, and F is the rate of latitudinal heat flow per unit area. T is temperature.

system is extremized subject to the energy balance constraints. The extremization proceeds as follows:

$$\frac{\partial}{\partial T_0}\left(F/T_1 - F/T_0\right) = 0 \tag{10.4}$$

Using Equations 10.1, 10.2, and 10.3 this becomes:

$$\frac{\partial}{\partial T_0}\left(\frac{I_0 - A - BT_0}{T_1} - \frac{I_0 - A - BT_0}{T_0}\right) = 0 \tag{10.5}$$

The sum of Equations 10.1 and 10.2 is:

$$I_0 + I_1 = 2A + B\left(T_0 + T_1\right) \tag{10.6}$$

Using Equation 10.6 to eliminate T_1, Equation 10.5 becomes:

$$\frac{\partial}{\partial T_0}\left(\frac{I_0 - A - BT_0}{\frac{I_0+I_1-2A}{B} - T_0} - \frac{I_0 - A - BT_0}{T_0}\right) = 0 \tag{10.7}$$

After differentiation, Equation 10.7 becomes:

$$\frac{-B}{\frac{I_0+I_1-2A}{B} - T_0} + \frac{I_0 - A - BT_0}{\left(\frac{I_0+I_1-2A}{B} - T_0\right)^2} + \frac{I_0 - A}{T_0^2} = 0 \tag{10.8}$$

Using Equation 10.6 to reintroduce T_1, Equation 10.8 becomes:

$$\frac{-B}{T_1} + \frac{I_0 - A - BT_0}{T_1^2} + \frac{I_0 - A}{T_0^2} = 0 \tag{10.9}$$

Using Equation 10.6 again to reintroduce I_1, this becomes:

$$\frac{-B}{T_1} - \frac{I_1 - A - BT_1}{T_1^2} + \frac{I_0 - A}{T_0^2} = 0 \tag{10.10}$$

Rearranging:

$$\frac{I_0 - A}{T_0^2} = \frac{I_1 - A}{T_1^2} \tag{10.11}$$

Solving for T_0:

$$T_0 = T_1 \left(\frac{I_0 - A}{I_1 - A} \right)^{1/2} \tag{10.12}$$

This result can be used with Equation 10.6 to solve for T_0 and T_1 using the known I_0, I_1, A, and B.

10.2.2 Solution for F

Simple climate models often try to predict F and the following work investigates this model's predictions. F is often parameterized as being proportional to the temperature difference between the two boxes.

$$F = 2D\,(T_0 - T_1) \tag{10.13}$$

D is a coefficient of heat transport, taken as an unknown to be solved for. Predicting D is equivalent to predicting F. The factor of two is included for consistency with other models (Lorenz et al., 2001).

I use Equations 10.13 and 10.3 to replace F and E_i in Equations 10.1 and 10.2, then substitute the result of Equation 10.12 for T_0 into Equations 10.1 and 10.2:

$$I_0 - 2DT_1\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) - A - BT_1\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} = 0 \tag{10.14}$$

$$I_1 + 2DT_1\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) - A - BT_1 = 0 \tag{10.15}$$

Rearranged, Equation 10.14 becomes:

$$(I_0 - A) = T_1\left(2D\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) + B\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2}\right) \tag{10.16}$$

Using Equation 10.16 to eliminate T_1 from Equation 10.15 shows that:

$$\begin{aligned}(I_1 - A) + \frac{2D\,(I_0 - A)\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right)}{2D\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) + B\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2}} - \\ \frac{B\,(I_0 - A)}{2D\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) + B\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2}} = 0\end{aligned} \tag{10.17}$$

Rearranging:

$$\begin{aligned}(I_1 - A)\,2D\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) + B\,(I_0 - A)^{1/2}\,(I_1 - A)^{1/2} + \\ (I_0 - A)\,2D\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) - B\,(I_0 - A) = 0\end{aligned} \tag{10.18}$$

Rearranging to group together like terms:

$$\begin{aligned}2D\,((I_0 - A) + (I_1 - A))\left(\left(\frac{I_0 - A}{I_1 - A}\right)^{1/2} - 1\right) = \\ B\,(I_0 - A)^{1/2}\left((I_0 - A)^{1/2} - (I_1 - A)^{1/2}\right)\end{aligned} \tag{10.19}$$

Multiplying by $(I_1 - A)^{1/2}$:

$$2D\left((I_0 - A) + (I_1 - A)\right)\left((I_0 - A)^{1/2} - (I_1 - A)^{1/2}\right) = \\ B\,(I_0 - A)^{1/2}\,(I_1 - A)^{1/2}\left((I_0 - A)^{1/2} - (I_1 - A)^{1/2}\right) \tag{10.20}$$

Finally solving for D:

$$\frac{2D}{B} = \frac{(I_0 - A)^{1/2}\,(I_1 - A)^{1/2}}{(I_0 - A) + (I_1 - A)} \tag{10.21}$$

For the special case where $I_0 = I_1$, Equation 10.21 states that $4D = B$. If $I_0 = I - \delta/2$ and $I_1 = I + \delta/2$, then Equation 10.21 becomes:

$$\frac{4D}{B} = \frac{(I - A + \delta/2)^{1/2}\,(I - A - \delta/2)^{1/2}}{(I - A)} \tag{10.22}$$

Rearranging:

$$\frac{4D}{B} = \left(1 - \frac{\delta^2}{4\,(I - A)^2}\right)^{1/2} \tag{10.23}$$

As long as $I - A$ is not much smaller than I, the value of this expression is close to unity. If I use a linearized Stefan-Boltzman law for E, then:

$$-3\sigma T^4 + 4\sigma T^3 T_i = E_i \\ A + BT_i = E_i \tag{10.24}$$

Where T is a characteristic temperature at the centre of the linear expansion. If T_0 and T_1 are not too dissimilar from T, then $A \sim -3 \times E_i$. $I - A$ is not much smaller than I, in fact it is significantly larger than I. If $E(T)$ is any polynomial function of greater than first order that monotonically increases with T, then its linearized form has A negative. This leads to a value for Equation 10.23 that is not too far from unity. As an example, if $A = 0$ and $\delta = I$, then Equation 10.23 has a value of 0.87. Smaller values of δ are closer to unity. To a reasonable approximation:

$$4D = B \tag{10.25}$$

In investigating this simple climate model, I have discovered a conserved quantity that has the same value in each box (Equation 10.11) and a relationship between the coefficient of heat transport within the atmosphere (D) and the derivative of outgoing radiation flux with respect to temperature (B) (Equation 10.25). These properties can be used to make predictions and test the model.

10.3 Other Simple Climate Models

In Section 10.3 I investigate three other simple climate models that have been proposed in the literature (Paltridge, 1975; Rodgers, 1976; Lin, 1982). These three models all use different variational principles to predict the annually-averaged variation of temperature with latitude. I show that their predictions are almost identical in most cases.

Steady state, one dimensional climate models are generally of the form:

$$F(x) = I(x) - E(x) \tag{10.26}$$

where x is the sine of latitude, F is the meridional heat flux, I is the absorbed solar radiation, and E is the emitted thermal radiation. The effective temperature, T, is defined by:

$$E = \sigma T^4 \tag{10.27}$$

All temperatures discussed in Section 10.3 are effective temperatures, so I have dropped the usual subscript-e for convenience. I depends on planetary orbital parameters and Bond albedo and F or E is to be predicted by a climate model. Energy balance requires that:

$$\int_{-1}^{1} I(x) - E(x)dx = 0 \tag{10.28}$$

In the trivial case where there is no heat transport, $E(x) = I(x)$ and $T(x)$ is easily predicted. In the trivial case where there is infinite ability to transport heat, $E(x)$ is constant and equal to the mean of $I(x)$; $T(x)$ is again easily predicted.

I now introduce some useful notation:

$$\begin{aligned} < y >= \frac{\int_{-1}^{1} y(x)dx}{\int_{-1}^{1} dx} \\ < y >= \frac{\int_{-1}^{1} y(x)dx}{2} \end{aligned} \tag{10.29}$$

10.3.1 Rodgers's Model

Rodgers (1976), in response to Paltridge (1975), discussed a simple model in which $E(x)$ is that which extremizes:

$$\int_{-1}^{1} \frac{I(x) - E(x)}{E(x)} dx \tag{10.30}$$

Using the calculus of variations and the energy balance constraint of Equation 10.28, the solution to this model is given by (Arfken and Weber, 1995):

$$\frac{\partial}{\partial E}\left(I(x)/E(x) - 1 - \lambda I(x) + \lambda E(x)\right) = 0 \tag{10.31}$$

Where λ is a Lagrange multiplier. Performing the differentiation in Equation 10.31 shows that:

$$\frac{-I(x)}{E(x)^2} + \lambda = 0 \tag{10.32}$$

The solution of Equation 10.32 for $E(x)$ is:

$$\left(\frac{I(x)}{\lambda}\right)^{1/2} = E(x) \tag{10.33}$$

Substituting the result of Equation 10.33 into Equation 10.28 provides a solution for λ:

$$\int_{-1}^{1} I(x)dx = \int_{-1}^{1} \left(\frac{I(x)}{\lambda}\right)^{1/2} dx \tag{10.34}$$

Using Equation 10.29, Equation 10.34 becomes:

$$< I > = \frac{< I^{1/2} >}{\lambda^{1/2}} \tag{10.35}$$

Using Equation 10.35 to eliminate λ from Equation 10.33 gives the following solution for $E(x)$:

$$\sigma T(x)^4 = E(x) = I(x)^{1/2} \times \frac{< I >}{< I^{1/2} >} \tag{10.36}$$

10.3.2 Paltridge's Model

Paltridge (1975) proposed a simple model in which $T(x)$ is that which extremizes:

$$\int_{-1}^{1} \frac{I(x) - \sigma T^4(x)}{T(x)} dx \tag{10.37}$$

Subject to the energy balance constraint of Equation 10.28. Using the calculus of variations, the solution of Equation 10.37 is:

$$\frac{\partial}{\partial T}\left(I(x)/T(x) - \sigma T(x)^3 - \lambda I(x) + \lambda\sigma T(x)^4\right) = 0 \tag{10.38}$$

Performing the differentiation in Equation 10.38 shows that:

$$\frac{-I(x)}{T(x)^2} - 3\sigma T(x)^2 + 4\lambda\sigma T(x)^3 = 0 \tag{10.39}$$

Rearranging:

$$I(x) + 3\sigma T(x)^4 = 4\lambda\sigma T(x)^5 \tag{10.40}$$

Integrating Equation 10.40 and using Equation 10.28 to replace the σT^4 term, I find:

$$4 < I >= 4\lambda\sigma < T^5 > \tag{10.41}$$

Substituting this solution for λ into Equation 10.40 shows that:

$$I(x) + 3\sigma T(x)^4 = 4\frac{< I >}{< T^5 >}T(x)^5 \tag{10.42}$$

I have not found a simple solution for $T(x)$, so I leave the solution in this form and move on.

10.3.3 Lin's Model

Lin (1982) proposed a simple model in which the generation of available potential energy is maximized. I adapt his model by replacing his use of surface temperature with effective temperature. This makes it possible for his model to be compared with those of Rodgers and Paltridge. In my adaption of Lin's model $T(x)$ is that which extremizes:

$$\int_{-1}^{1} \left(I(x) - \sigma T(x)^4\right)(T(x) - < T >)\, dx \tag{10.43}$$

Subject to the energy balance constraint of Equation 10.28.

Using the calculus of variations, the solution of Equation 10.43 is:

$$\begin{aligned}\frac{\partial}{\partial T}\left(I(x)T(x) - \sigma T(x)^5 - I(x) < T >\right) + \\ \frac{\partial}{\partial T}\left(\sigma < T > T(x)^4 - \lambda I(x) + \lambda\sigma T(x)^4\right) = 0\end{aligned} \tag{10.44}$$

Performing the differentiation in Equation 10.44 shows that:

$$I(x) - 5\sigma T(x)^4 + 4\sigma < T > T(x)^3 + 4\lambda\sigma T(x)^3 = 0 \tag{10.45}$$

Integrating Equation 10.45 and using Equation 10.28 to replace the σT^4 term, I find:

$$< I > -5 < I > +4\sigma < T >< T^3 > +4\lambda\sigma < T^3 >= 0 \tag{10.46}$$

Rearranging:

$$\sigma < T >< T^3 > +\lambda\sigma < T^3 >=< I > \tag{10.47}$$

Rearranging to solve for λ:

$$\lambda = \frac{< I >}{\sigma < T^3 >} - < T > \tag{10.48}$$

Substituting the solution for λ from Equation 10.48 into Equation 10.45, I find:

$$\begin{aligned} I(x) - 5\sigma T(x)^4 + 4\sigma < T > T(x)^3 - \\ 4\sigma < T > T(x)^3 + \frac{4\sigma T(x)^3 < I >}{\sigma < T^3 >} = 0 \end{aligned} \tag{10.49}$$

Rearranging:

$$I(x) - 5\sigma T(x)^4 + \frac{4\sigma T(x)^3 < I >}{\sigma < T^3 >} = 0 \tag{10.50}$$

I have not found a simple solution for $T(x)$, so I leave the solution in this form and move on.

10.3.4 Restricted Solution

To investigate the similarity of this three models, I assume that $I(x)$ is symmetrical about the equator and close to uniform:

$$I(x) = I_0 + ax^2 \tag{10.51}$$

Where $a/I_0 \ll 1$. This could be rewritten to use Legendre polynomials if desired. The omission of a term linear in x is equivalent to assuming a circular orbit and hemispherically symmetric albedo. Using Equation 10.51, $< I >$ becomes:

$$\begin{gathered} < I >= \frac{1}{2}\int_{-1}^{1} I_0 + ax^2 dx \\ < I >= I_0 + \frac{a}{3} \end{gathered} \tag{10.52}$$

I now find the solution for $E(x)$ for each of the three models using this $I(x)$.

10.3.5 Restricted Solution to Rodgers's Model

Equation 10.36 relates $E(x)$ to $I(x)$ in Rodgers's model. To find $E(x)$ using $I(x)$ from Equation 10.51, I first find $I(x)^{1/2}$ and then $< I^{1/2} >$.

$$\begin{gathered} I(x)^{1/2} = \left(I_0 + ax^2\right)^{1/2} \\ I(x)^{1/2} = I_0^{1/2}\left(1 + \frac{ax^2}{2I_0}\right) \end{gathered} \tag{10.53}$$

$$\begin{gathered} < I^{1/2} >= \frac{1}{2}\int_{-1}^{1} I_0^{1/2}\left(1 + \frac{ax^2}{2I_0}\right) dx \\ < I^{1/2} >= I_0^{1/2}\left(1 + \frac{a}{6I_0}\right) \end{gathered} \tag{10.54}$$

Substituting the results of Equations 10.53 and 10.54 into Equation 10.36 I find:

$$E(x) = I_0^{1/2}\left(1 + \frac{ax^2}{2I_0}\right)\frac{\left(I_0 + \frac{a}{3}\right)}{I_0^{1/2}\left(1 + \frac{a}{6I_0}\right)} \tag{10.55}$$

Rearranging:

$$E(x) = I_0\left(1 + \frac{ax^2}{2I_0}\right)\left(1 + \frac{a}{3I_0}\right)\left(1 - \frac{a}{6I_0}\right) \tag{10.56}$$

Rearranging:

$$E(x) = I_0\left(1 + \frac{ax^2}{2I_0} + \frac{a}{3I_0} - \frac{a}{6I_0}\right) \tag{10.57}$$

Rearranging:

$$\sigma T(x)^4 = E(x) = I_0 + \frac{ax^2}{2} + \frac{a}{6} \tag{10.58}$$

10.3.6 Restricted Solution to Paltridge's Model

Equation 10.42 is not a simple solution for $E(x)$. Following the form of the restricted solution to Rodgers's model, Equation 10.58, I search for a solution of the form:

$$T(x) = T_0 + bx^2 + c \tag{10.59}$$

Where $b/T_0 \ll 1$ and $c/T_0 \ll 1$. Using Equation 10.59, I have:

$$T(x)^4 = {T_0}^4 + 4bx^2T_0^3 + 4cT_0^3 \tag{10.60}$$

$$T(x)^5 = {T_0}^5 + 5bx^2T_0^4 + 5cT_0^4 \tag{10.61}$$

$$< T^5 >= {T_0}^5 + \frac{5bT_0^4}{3} + 5cT_0^4 \tag{10.62}$$

Taking Equation 10.42 and substituting for $I(x)$ using Equation 10.51, for $< I >$ using Equation 10.52, for $T(x)^4$ using Equation 10.60, for $T(x)^5$ using Equation 10.61, and for $< T^5 >$ using Equation 10.62 I find:

$$\begin{aligned} I_0 + ax^2 + 3\sigma\left({T_0}^4 + 4bx^2T_0^3 + 4cT_0^3\right) = \\ 4\left(I_0 + \frac{a}{3}\right)\frac{{T_0}^5 + 5bx^2T_0^4 + 5cT_0^4}{{T_0}^5 + \frac{5bT_0^4}{3} + 5cT_0^4} \end{aligned} \tag{10.63}$$

Rearranging:

$$\begin{aligned} I_0 + ax^2 + 3\sigma\left({T_0}^4 + 4bx^2T_0^3 + 4cT_0^3\right) = \\ 4I_0\left(1 + \frac{a}{3I_0}\right)\frac{1 + \frac{5bx^2}{T_0} + \frac{5c}{T_0}}{1 + \frac{5b}{3T_0} + \frac{5c}{T_0}} \end{aligned} \tag{10.64}$$

Rearranging:

$$\begin{aligned} I_0 + ax^2 + 3\sigma{T_0}^4 + 12\sigma bx^2T_0^3 + 12\sigma cT_0^3 = \\ 4I_0\left(1 + \frac{a}{3I_0} + \frac{5bx^2}{T_0} + \frac{5c}{T_0} - \frac{5b}{3T_0} - \frac{5c}{T_0}\right) \end{aligned} \tag{10.65}$$

Rearranging:

$$I_0 + ax^2 + 3\sigma {T_0}^4 + 12\sigma bx^2T_0^3 + 12\sigma cT_0^3 = \tag{10.66}$$
$$4I_0 + \frac{4a}{3} + \frac{20bx^2I_0}{T_0} - \frac{20bI_0}{3T_0}$$

Grouping together large and uniform terms, small and non-uniform terms, and small and uniform terms, I have the following three equations:

$$I_0 + 3\sigma {T_0}^4 = 4I_0 \tag{10.67}$$

$$ax^2 + 12\sigma bx^2T_0^3 = \frac{20bx^2I_0}{T_0} \tag{10.68}$$

$$12\sigma cT_0^3 = \frac{4a}{3} - \frac{20bI_0}{3T_0} \tag{10.69}$$

Equation 10.67 shows that:

$$\sigma {T_0}^4 = I_0 \tag{10.70}$$

Using the result of Equation 10.70 to eliminate I_0 from Equations 10.68 and 10.69, I have the following two equations:

$$ax^2 + 12\sigma bx^2T_0^3 = 20\sigma bx^2T_0^3 \tag{10.71}$$

$$12\sigma c T_0^3 = \frac{4a}{3} - \frac{20\sigma b x^2 T_0^3}{3} \tag{10.72}$$

Equation 10.71 shows that the solution for b is:

$$a/8 = \sigma b T_0^3 \tag{10.73}$$

Substituting the solution of Equation 10.73 for b into Equation 10.72, I have:

$$\begin{aligned} 12\sigma c T_0^3 &= \frac{4a}{3} - \frac{20a}{3 \times 8} = \frac{a}{2} \\ \sigma c T_0^3 &= \frac{a}{24} \end{aligned} \tag{10.74}$$

Substituting the results of Equations 10.70, 10.73 and 10.74 into Equation 10.60 and multiplying by σ, I have:

$$\begin{aligned} E(x) = \sigma T(x)^4 &= \sigma {T_0}^4 + 4\sigma b x^2 T_0^3 + 4\sigma c T_0^3 \\ E(x) &= I_0 + \frac{ax^2}{2} + \frac{a}{6} \end{aligned} \tag{10.75}$$

This restricted solution for $E(x)$ in Paltridge's model is the same as in Rodgers's model (Equation 10.58).

10.3.7 Restricted Solution to Lin's Model

Equation 10.50 is not a simple solution for $E(x)$. Following the form of the restricted solution to Rodgers's model, Equation 10.58, I search for a solution of the form:

$$T(x) = T_0 + bx^2 + c \tag{10.76}$$

Where $b/T_0 \ll 1$ and $c/T_0 \ll 1$. Using Equation 10.76, I have:

$$T(x)^4 = T_0{}^4 + 4bx^2T_0^3 + 4cT_0^3 \tag{10.77}$$

$$T(x)^3 = T_0{}^3 + 3bx^2T_0^2 + 3cT_0^2 \tag{10.78}$$

$$< T^3 >= T_0{}^3 + bT_0^2 + 3cT_0^2 \tag{10.79}$$

Taking Equation 10.50 and substituting for $I(x)$ using Equation 10.51, for $< I >$ using Equation 10.52, for $T(x)^4$ using Equation 10.77, for $T(x)^3$ using Equation 10.78, and for $< T^3 >$ using Equation 10.79 I find:

$$\begin{aligned} I_0 + ax^2 - 5\sigma\left(T_0{}^4 + 4bx^2T_0^3 + 4cT_0^3\right) + \\ 4\left(I_0 + \frac{a}{3}\right)\frac{T_0{}^3 + 3bx^2T_0^2 + 3cT_0^2}{T_0{}^3 + bT_0^2 + 3cT_0^2} = 0 \end{aligned} \tag{10.80}$$

Rearranging:

$$\begin{aligned} I_0 + ax^2 - 5\sigma T_0{}^4 - 20\sigma bx^2T_0^3 - 20\sigma cT_0^3 + \\ 4I_0\left(1 + \frac{a}{3I_0}\right)\frac{1 + \frac{3bx^2}{T_0} + \frac{3c}{T_0}}{1 + \frac{b}{T_0} + \frac{3c}{T_0}} = 0 \end{aligned} \tag{10.81}$$

Rearranging:

$$I_0 + ax^2 - 5\sigma {T_0}^4 - 20\sigma bx^2T_0^3 - 20\sigma cT_0^3 + \\ 4I_0\left(1 + \frac{a}{3I_0} + \frac{3bx^2}{T_0} + \frac{3c}{T_0} - \frac{b}{T_0} - \frac{3c}{T_0}\right) = 0 \tag{10.82}$$

Rearranging:

$$I_0 + ax^2 - 5\sigma {T_0}^4 - 20\sigma bx^2T_0^3 - 20\sigma cT_0^3 + \\ 4I_0 + \frac{4a}{3} + \frac{12bx^2I_0}{T_0} - \frac{4bI_0}{T_0} = 0 \tag{10.83}$$

Grouping together large and uniform terms, small and non-uniform terms, and small and uniform terms, I have the following three equations:

$$I_0 - 5\sigma {T_0}^4 + 4I_0 = 0 \tag{10.84}$$
$$\sigma {T_0}^4 = I_0$$

$$ax^2 - 20\sigma bx^2T_0^3 + \frac{12bx^2I_0}{T_0} = 0 \tag{10.85}$$

$$-20\sigma cT_0^3 + \frac{4a}{3} - \frac{4bI_0}{T_0} = 0 \tag{10.86}$$

Using the result of Equation 10.84 to eliminate I_0 from Equations 10.85 and 10.86, I have the following two equations:

$$ax^2 - 20\sigma bx^2T_0^3 + 12\sigma bx^2T_0^3 = 0 \tag{10.87}$$
$$a/8 = \sigma bT_0^3$$

$$-20\sigma c T_0^3 + \frac{4a}{3} - 4\sigma b x^2 T_0^3 = 0 \tag{10.88}$$

Substituting the solution of Equation 10.87 for b into Equation 10.88, I have:

$$\begin{aligned} -20\sigma c T_0^3 + \frac{4a}{3} - \frac{a}{2} &= 0 \\ \frac{5a}{6} &= 20\sigma c T_0^3 \\ \sigma c T_0^3 &= \frac{a}{24} \end{aligned} \tag{10.89}$$

Substituting the results of Equations 10.84, 10.87 and 10.90 into Equation 10.77 and multiplying by σ, I have:

$$\begin{aligned} E(x) = \sigma T(x)^4 &= \sigma {T_0}^4 + 4\sigma b x^2 T_0^3 + 4\sigma c T_0^3 \\ E(x) &= I_0 + \frac{ax^2}{2} + \frac{a}{6} \end{aligned} \tag{10.90}$$

This restricted solution for $E(x)$ in Lin's model is the same as in Rodgers's model (Equation 10.58).

10.3.8 Restricted Solution to Zero and Infinite Heat Transport Models

In the limit in which there is no heat transport within the atmosphere, $E(x) = I(x)$ and Equation 10.51 shows that:

$$E(x) = I_0 + ax^2 \tag{10.91}$$

In the limit in which there is infinite ability to transport heat within the atmosphere, $E(x) =< I >$ and Equation 10.52 shows that:

$$E(x) = I_0 + \frac{a}{3} \tag{10.92}$$

The mean of these two limits for $E(x)$ is:

$$E(x) = I_0 + \frac{ax^2}{2} + \frac{a}{6} \tag{10.93}$$

This expression for $E(x)$ is *also* the same as the restricted solution for $E(x)$ in Rodgers's model (Equation 10.58).

10.3.9 Conclusions

In the limit $I_0 + ax^2$, where $a/I_0 \ll 1$, Rodgers's, Paltridge's, and Lin's models all have the same solution for $E(x)$. This solution is also the mean of models with zero and infinite ability to transport heat. I have not found the solutions of next highest order in x^2 for $E(x)$ for the three models. I have not experimented with numerical values in these equations to see how large the x^4 term in $I(x)$ can be before these three models make predictions for $E(x)$ that differ by, say, 50%. Since $T(x) \propto E(x)^{1/4}$, this corresponds to predictions for $T(x)$ that differ by only 10%. It can be seen that these three models give very similar predictions for $T(x)$ for quite large deviations of $I(x)$ from being uniform in x.

Since these three models make nearly-identical predictions, testing one of them is almost equivalent to testing all of them. It will be very difficult to prove that one is true and the other two are false.