APPENDIX F

UNCERTAINTY ANALYSES FOR SECTION 6.3

F.1 Uncertainties for Section 6.3.1

Using Equation 6.33:

$$\left(\frac{\sigma_{\rho_{n+1/2}}}{\rho_{n+1/2}}\right)^2 = \left(\frac{\sigma_{a_{n+1/2}}}{a_{n+1/2}}\right)^2 + \left(\frac{\sigma_{v^2_{n+1/2}}}{v^2_{n+1/2}}\right)^2 \tag{F.1}$$

$$\left(\frac{\sigma_{\rho_{n+1/2}}}{\rho_{n+1/2}}\right)^2 = \left(\frac{\sigma_{a_{n+1/2}}}{a_{n+1/2}}\right)^2 + \left(\frac{2\sigma_{v_{n+1/2}}}{v_{n+1/2}}\right)^2 \tag{F.2}$$

$$\left(\frac{\sigma_{\rho_{n+1/2}}}{\rho_{n+1/2}}\right)^2 = \frac{\sigma^2_{a_{n+1/2}}}{a^2_{n+1/2}} + \frac{4\sigma^2_{v_{n+1/2}}}{v^2_{n+1/2}} \tag{F.3}$$

Using Equations 6.14, 6.15, 6.16, and 6.17:

$$\left(\frac{\sigma_{\rho_{n+1/2}}}{\rho_{n+1/2}}\right)^2 = \frac{\left(\frac{\sqrt{2}\sigma_v}{\Delta}\right)^2}{\left(\frac{v_{n+1}-v_n}{\Delta}\right)^2} + 4\frac{\left(\frac{\sigma_v}{\sqrt{2}}\right)^2}{\left(\frac{v_n+v_{n+1}}{2}\right)^2} \tag{F.4}$$

$$\left(\frac{\sigma_{\rho_{n+1/2}}}{\rho_{n+1/2}}\right)^2 = \frac{2\sigma_v^2}{(v_{n+1}-v_n)^2} + \frac{8\sigma_v^2}{(v_n+v_{n+1})^2} \tag{F.5}$$

Using Equation 6.34:

$$\sigma^2_{\rho_{n+1/2}} = \left(\frac{v_{n+1} - v_n}{(v_n + v_{n+1})^2}\frac{4m}{\Delta A C_D}\right)^2 \sigma_v^2 \left(\frac{2}{(v_{n+1} - v_n)^2} + \frac{8}{(v_n + v_{n+1})^2}\right) \tag{F.6}$$

$$\sigma^2_{\rho_{n+1/2}} = \left(\frac{4m}{\Delta A C_D}\right)^2 \frac{\sigma_v^2}{(v_n + v_{n+1})^6}\left(2\,(v_n + v_{n+1})^2 + 8\,(v_{n+1} - v_n)^2\right) \tag{F.7}$$

$$\sigma_{\rho_{n+1/2}} = \left(\frac{4m}{\Delta A C_D}\right) \frac{\sigma_v}{(v_n + v_{n+1})^3}\left(2\,(v_n + v_{n+1})^2 + 8\,(v_{n+1} - v_n)^2\right)^{1/2} \tag{F.8}$$

F.2 Uncertainties for Section 6.3.2

Using Equation 6.36:

$$\sigma^2_{p^*_{n+1}} = \sigma^2_{p^*_n} + g^2\left((z_n - z_{n+1})^2\,\sigma^2_{\rho_{n+1/2}} + \rho^2_{n+1/2}\sigma^2_{z_n - z_{n+1}}\right) \tag{F.9}$$

$\sigma_{\rho_{n+1/2}}$ can be calculated from Equation F.8 and $\sigma_{z_n - z_{n+1}}$ can be calculated from Equation 6.13. $\rho_{n+1/2}$ can be calculated from Equation 6.33 and $(z_n - z_{n+1})$ can be calculated from Equation 6.8.

F.3 Uncertainties for Section 6.3.3

Using Equation 6.38:

$$\left(\sigma_{\rho_n^2}\right)^2 = \rho_n^4\frac{\sigma^2_{\rho_{n-1/2}}}{\rho^2_{n-1/2}} + \rho_n^4\frac{\sigma^2_{\rho_{n+1/2}}}{\rho^2_{n+1/2}} \tag{F.10}$$

Using Equation 6.38:

$$\left(\sigma_{\rho_n^2}\right)^2 = \rho_{n+1/2}^2 \sigma_{\rho_{n-1/2}}^2 + \rho_{n-1/2}^2 \sigma_{\rho_{n+1/2}}^2 \quad \text{(F.11)}$$

$$\frac{\sigma_{\rho_n}}{\rho_n} = \frac{2\sigma_{\rho_n^2}}{\rho_n^2} \quad \text{(F.12)}$$

$$\sigma_{\rho_n} = \frac{2}{\sqrt{\rho_{n-1/2}\rho_{n+1/2}}} \left(\rho_{n+1/2}^2 \sigma_{\rho_{n-1/2}}^2 + \rho_{n-1/2}^2 \sigma_{\rho_{n+1/2}}^2\right)^{1/2} \quad \text{(F.13)}$$

This can be calculated using Equations F.8 and 6.34. Using Equation 6.39:

$$\left(\frac{\sigma_{T_n^*}}{T_n^*}\right)^2 = \left(\frac{\sigma_{p_n^*}}{p_n^*}\right)^2 + \left(\frac{\sigma_{\rho_n}}{\rho_n}\right)^2 \quad \text{(F.14)}$$

$$\sigma_{T_n^*}^2 = \frac{T_n^{*2}}{p_n^{*2}} \sigma_{p_n^*}^2 + \frac{T_n^{*2}}{\rho_n^2} \sigma_{\rho_n}^2 \quad \text{(F.15)}$$

$$\sigma_{T_n^*}^2 = \left(\frac{M_{mol}}{k_B}\right)^2 \left(\frac{\sigma_{p_n^*}^2}{\rho_n^2} + \frac{p_n^{*2}\sigma_{\rho_n}^2}{\rho_n^4}\right) \quad \text{(F.16)}$$

$\sigma_{p_n^*}$ can be calculated from Equation F.9 and σ_{ρ_n} can be calculated from Equation F.13. p_n^* can be calculated from Equation 6.36 and ρ_n can be calculated from Equation 6.38.

F.4 Uncertainties for Section 6.3.4

The uncertainty calculation is quite involved. Since I did not find a simple expression for $\sigma_{T_n^@}$, I outline a scheme for calculating numerically what $\sigma_{T_n^@}$ is:

Define P as:

$$P = v_{n-1} + 6v_n + v_{n+1} \tag{F.17}$$

Define Q as:

$$Q = \ln\left(\frac{v_{n+1} - v_n}{(v_n + v_{n+1})^2}\frac{(v_{n-1} + v_n)^2}{v_n - v_{n-1}}\right) \tag{F.18}$$

Using Equation 6.45:

$$T_n^@ = \frac{M_{mol}g}{8k_B}\Delta\frac{P}{Q} \tag{F.19}$$

$$\left(\frac{\sigma_{T_n^@}}{T_n^@}\right)^2 = \left(\frac{\sigma_P}{P}\right)^2 + \left(\frac{\sigma_Q}{Q}\right)^2 \tag{F.20}$$

$$\sigma_P^2 = \sigma_v^2 + 36\sigma_v^2 + \sigma_v^2 \tag{F.21}$$

$$\sigma_P = \sigma_v\sqrt{38} \tag{F.22}$$

Define R as:

$$R = \left(\frac{v_{n+1} - v_n}{(v_n + v_{n+1})^2} \frac{(v_{n-1} + v_n)^2}{v_n - v_{n-1}} \right) \tag{F.23}$$

Hence:

$$Q = \ln R \tag{F.24}$$

$$\sigma_Q = \frac{\sigma_R}{R} \tag{F.25}$$

Define S as:

$$S = v_{n+1} - v_n \tag{F.26}$$

Define Z as:

$$Z = (v_n + v_{n+1})^2 \tag{F.27}$$

Define U as:

$$U = (v_{n-1} + v_n)^2 \tag{F.28}$$

Define W as:

$$W = v_n - v_{n-1} \tag{F.29}$$

Hence:

$$R = \frac{S}{Z}\frac{U}{W} \tag{F.30}$$

$$\left(\frac{\sigma_R}{R}\right)^2 = \left(\frac{\sigma_S}{S}\right)^2 + \left(\frac{\sigma_Z}{Z}\right)^2 + \left(\frac{\sigma_U}{U}\right)^2 + \left(\frac{\sigma_W}{W}\right)^2 \tag{F.31}$$

$$\sigma_S^2 = \sigma_W^2 = 2\sigma_v^2 \tag{F.32}$$

Define X as:

$$X = (v_n + v_{n+1}) \tag{F.33}$$

Hence:

$$Z = X^2 \tag{F.34}$$

$$\frac{\sigma_Z}{Z} = \frac{2\sigma_X}{X} \tag{F.35}$$

$$\sigma_X = \sqrt{2}\sigma_v \tag{F.36}$$

Define Y as:

$$Y = (v_{n-1} + v_n) \tag{F.37}$$

Hence:

$$U = Y^2 \tag{F.38}$$

$$\frac{\sigma_U}{U} = \frac{2\sigma_Y}{Y} \tag{F.39}$$

$$\sigma_Y = \sqrt{2}\sigma_v \tag{F.40}$$

F.5 Uncertainties for Section 6.3.5

Using Equation 6.46:

$$\frac{\sigma^2_{p^@_n}}{p^{@2}_n} = \frac{\sigma^2_{\rho_n}}{\rho^2_n} + \frac{\sigma^2_{T^@_n}}{T^{@2}_n} \tag{F.41}$$

σ_{ρ_n} can be calculated using Equation F.13 and $\sigma_{T^@_n}$ can be calculated using Equation F.20. $T^@_n$ can be calculated using Equation 6.45 and ρ_n can be calculated from Equation 6.38.

F.6 Uncertainties for Section 6.3.6

Using Equation 6.51:

$$\left(\frac{\sigma_{T^\#_n}}{T^\#_n}\right)^2 = \left(\frac{\sigma_{v^2}}{v^2_n}\right)^2 + \left(\frac{\sigma_{v_{n-1}-v_{n+1}}}{v_{n-1} - v_{n+1}}\right)^2 \tag{F.42}$$

$$\left(\frac{\sigma_{T_n^{\#}}}{T_n^{\#}}\right)^2 = \left(\frac{2\sigma_v}{v_n}\right)^2 + \left(\frac{\sqrt{2}\sigma_v}{v_{n-1} - v_{n+1}}\right)^2 \tag{F.43}$$

$$\left(\frac{\sigma_{T_n^{\#}}}{T_n^{\#}}\right)^2 = \frac{4\sigma_v^2}{v_n^2} + \frac{2\sigma_v^2}{\left(v_{n-1} - v_{n+1}\right)^2} \tag{F.44}$$

Using Equation 6.51:

$$\sigma_{T_n^{\#}}^2 = \left(\frac{M_{mol} g \Delta}{k_B}\right)^2 \left(\frac{v_n^2}{v_{n-1} - v_{n+1}}\right)^2 \sigma_v^2 \left(\frac{4}{v_n^2} + \frac{2}{\left(v_{n-1} - v_{n+1}\right)^2}\right) \tag{F.45}$$

$$\sigma_{T_n^{\#}}^2 = \left(\frac{M_{mol} g \Delta}{k_B}\right)^2 \frac{v_n^2}{\left(v_{n-1} - v_{n+1}\right)^4} \sigma_v^2 \left(4\left(v_{n-1} - v_{n+1}\right)^2 + 2v_n^2\right) \tag{F.46}$$

This uncertainty is less than before (Equation F.20) since the expression for calculating T_n (Equation 6.51) is so much simpler than before (Equation 6.45). However, there are only restricted circumstances in which it can be used.

F.7 Uncertainties for Section 6.3.7

Using Equation 6.56:

$$\sigma_{p_n^{\#}} = \frac{mg}{C_D A} \frac{\sigma_v}{v_n} \tag{F.47}$$

F.8 Uncertainties for Section 6.3.8

The uncertainty calculation is quite involved. Since I did not find a simple expression for $\sigma_{T_n^{\$}}$, I outline a scheme for calculating numerically what $\sigma_{T_n^{\$}}$ is. Neglecting p_0 and using Equation 6.56:

$$\frac{p^{\#}_{n+1}}{p^{\#}_{n}} = \frac{\ln v_{n+1} - \ln v_x}{\ln v_n - \ln v_x} \tag{F.48}$$

Define B as:

$$B = v_n + v_{n+1} \tag{F.49}$$

Define C as:

$$C = \ln\left(\frac{p^{\#}_{n+1}}{p^{\#}_{n}}\right) \tag{F.50}$$

Hence:

$$T^{\$}_{n+1/2} = \frac{M_{mol} g \Delta}{2k_B} \frac{B}{C} \tag{F.51}$$

$$\left(\frac{\sigma_{T^{\$}_{n+1/2}}}{T^{\$}_{n+1/2}}\right)^2 = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_C}{C}\right)^2 \tag{F.52}$$

$$\sigma_B^2 = 2\sigma_v^2 \tag{F.53}$$

Using Equation F.48, I define D as:

$$D = \frac{p^{\#}_{n+1}}{p^{\#}_{n}} = \frac{\ln v_{n+1} - \ln v_x}{\ln v_n - \ln v_x} \tag{F.54}$$

Hence:

$$C = \ln D \tag{F.55}$$

$$\sigma_C = \frac{\sigma_D}{D} \tag{F.56}$$

Define E as:

$$E = \ln v_{n+1} - \ln v_x \tag{F.57}$$

Define F as:

$$F = \ln v_n - \ln v_x \tag{F.58}$$

Hence:

$$D = \frac{E}{F} \tag{F.59}$$

$$\left(\frac{\sigma_D}{D}\right)^2 = \left(\frac{\sigma_E}{E}\right)^2 + \left(\frac{\sigma_F}{F}\right)^2 \tag{F.60}$$

$$\sigma_E = \frac{\sigma_v}{v_{n+1}} \tag{F.61}$$

$$\sigma_F = \frac{\sigma_v}{v_n} \tag{F.62}$$