APPENDIX E

ESTIMATING THE IMPORTANCE OF LATITUDINAL CHANGES IN PRESSURE

It is useful to make a simple estimate of how important the effects of latitude are for influencing derived pressure profiles. Here I outline a technique for doing so.

The conservation of momentum equations for the quasi-cyclostrophic case are, as in Equations 3.40 – 3.41:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{GM}{r^2} \tag{E.1}$$

$$\frac{v_\phi^2}{r\tan\theta} = \frac{1}{\rho r}\frac{\partial p}{\partial \theta} - g_{eff,\theta} \tag{E.2}$$

The ideal gas equation of state is:

$$p = \rho\left(\frac{k_B}{M_{mol}}\right)T \tag{E.3}$$

I assume that $g_{eff,\theta}$ is negligible and that $GM/r^2 = g$ is uniform over the region of interest. I also assume that M_{mol} and T are uniform. Combining Equations E.1 and E.3 and the assumptions gives:

$$\frac{1}{p}\frac{\partial p}{\partial r} = \frac{\partial \ln p}{\partial r} = -\frac{1}{H} \tag{E.4}$$

where $H = k_B T / M_{mol} g$. I also assume that $\tan\theta$ is constant. This is a major assumption that is not always appropriate, but it makes generating a rough estimate for E much easier. I use t for my constant value of $\tan\theta$.

With those assumptions:

$$\ln p = \ln p_0 - \frac{(r - r_0)}{H} + \frac{(\theta - \theta_0)\, v_\phi^2}{gHt} \tag{E.5}$$

r and θ do not vary independently. The spacecraft must remain on the appropriate flight path through the atmosphere. This is shown in Figure E.1 where r_0 is the value of r at periapsis, θ_0 is the value of θ at periapsis, Δr is the difference between r at the top of the atmosphere and at periapsis, and $\Delta\theta$ is the difference between θ at the top of the atmosphere and at periapsis. The "top" of the atmosphere is where the density profile begins and ends. Since the size of the elliptical orbit is assumed to be much larger than the planet's radius, the flight path appears linear in this Figure.

$$r_0 = r\cos(\theta - \theta_0) \tag{E.6}$$

$$r - r_0 = r_0 \left(\frac{1}{\cos(\theta - \theta_0)} - 1 \right) \tag{E.7}$$

$$r - r_0 = r_0 \left(\frac{1}{1 - \frac{(\theta - \theta_0)^2}{2}} - 1 \right) \tag{E.8}$$

$$r - r_0 = r_0 \left(1 + \frac{(\theta - \theta_0)^2}{2} - 1 \right) \tag{E.9}$$

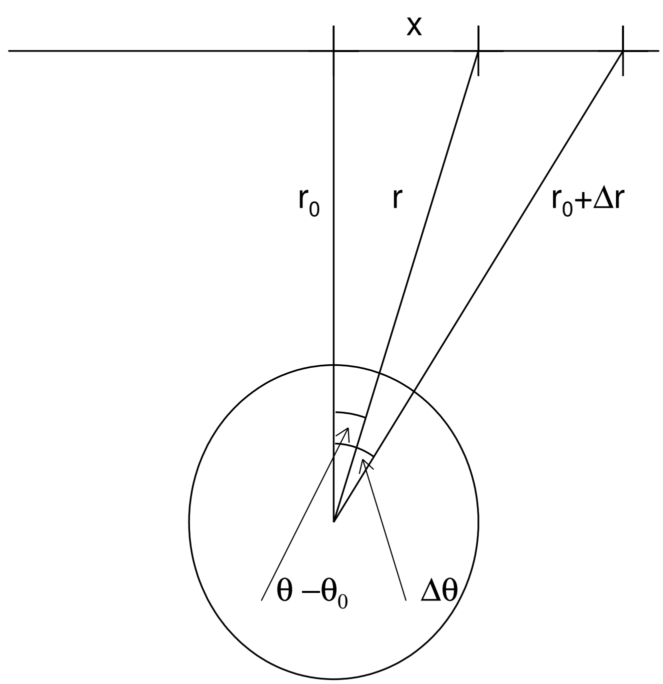

Figure E.1: Aerobraking Geometry

$$r - r_0 = \frac{r_0}{2}\left(\theta - \theta_0\right)^2 \tag{E.10}$$

I now introduce the superscript "+" to refer to the leg of the aerobraking pass with $\theta > \theta_0$. I later use the superscript "−" to refer to the other leg. The two legs correspond to the positive and negative square roots in Equation E.10. So flying along the positive leg of the flight path:

$$\ln p^+ = \ln p_0^+ - \frac{(r - r_0)}{H} + \frac{\sqrt{2}\left(r - r_0\right)^{1/2} v_\phi^2}{r_0^{1/2} g H t} \tag{E.11}$$

The only difference for the other leg is that the sign of the last term on the right hand side changes. Using gravity as the only force, the usual best estimate for periapsis pressure is:

$$p_{peri^+,est} = p_{top^+} + \int_{r_0}^{r_0+\Delta r} \rho^+ g dr \tag{E.12}$$

The subscript "est" refers to values that are calculated without any consideration of changes with latitude. The subscript "top" refers to values at the top of the atmosphere, either on inbound or outbound. ρ^+ is density measured along the inbound leg of the flight path. p_{top^+} can be related to the measured change in density with altitude (and latitude) at the top of the flight path.

With the earlier assumptions, density obeys a formula similar to Equation E.11:

$$\ln \rho^+ = \ln \rho_0^+ - \frac{(r - r_0)}{H} + \frac{\sqrt{2}\left(r - r_0\right)^{1/2} v_\phi^2}{r_0^{1/2} g H t} \tag{E.13}$$

At the top^{+} position on the flight path, $r = r_0 + \Delta r$ and $\theta = \theta_0 + \sqrt{2\Delta r / r_0}$, the density is:

$$\ln \rho = \ln \rho_0 - \frac{\Delta r}{H} + \frac{(2\Delta r)^{1/2} v_\phi^2}{r_0^{1/2} g H t} \tag{E.14}$$

At an altitude H below top^{+} and displaced in latitude to remain on the flight path, the density is:

$$\ln \rho = \ln \rho_0 - \frac{\Delta r}{H} + 1 + \frac{\sqrt{2\,(\Delta r - H)} v_\phi^2}{r_0^{1/2} g H t} \tag{E.15}$$

In practice, a scale height would be estimated from a best fit to all measured densities along the flight path between these two points. Here, I just estimate as if these are the only two data points. If the estimate for the scale height turns out to have a large influence on the results, then I will revise this. The estimated scale height from these two data points is $-\Delta z / \Delta \ln \rho$:

$$\frac{H}{1 - \frac{\sqrt{2\Delta r} v_\phi^2}{r_0^{1/2} g H t} \left(1 - (1 - H/\Delta r)^{1/2}\right)} \tag{E.16}$$

The estimated pressure is the product of ρ_{top+}, g, and this:

$$\frac{\rho_0 g H \left(\exp\left(\frac{-\Delta r}{H}\right)\right) \left(\exp\left(\frac{\sqrt{2\Delta r} v_\phi^2}{r_0^{1/2} g H t}\right)\right)}{1 - \frac{\sqrt{2\Delta r} v_\phi^2}{r_0^{1/2} g H t} \left(1 - (1 - H/\Delta r)^{1/2}\right)} \tag{E.17}$$

I now return to the other part of Equation E.12 — the integration.

$$p_{peri^+,est} = p_{top^+} + \int_{r_0}^{r_0+\Delta r} \rho_0 g \left(\exp\left(-\frac{r-r_0}{H}\right) \exp\left(\frac{\sqrt{2}\,(r-r_0)^{1/2} v_\phi^2}{r_0^{1/2} g H t}\right)\right) dr \quad \text{(E.18)}$$

Let:

$$x = \frac{\sqrt{2}\,(\Delta r)^{1/2} v_\phi^2}{r_0^{1/2} g H t} \quad \text{(E.19)}$$

Let:

$$y = \frac{H}{\Delta r} \quad \text{(E.20)}$$

Substituting x and y into Equation E.12 and redefining r to remove the $r - r_0$ terms:

$$p_{peri^+,est} = p_{top^+} + \rho_0 g \int_0^{\Delta r} \exp\left(-\frac{r}{H} + x\left(\frac{r}{\Delta r}\right)^{1/2}\right) dr \quad \text{(E.21)}$$

Let:

$$r' = \frac{r}{H} \quad \text{(E.22)}$$

Substituting r' into Equation E.21:

$$p_{peri^+,est} = p_{top^+} + \rho_0 g H \int_0^{\Delta r/H} \exp\left(-r' + x\left(\frac{Hr'}{\Delta r}\right)^{1/2}\right) dr' \quad \text{(E.23)}$$

Substituting y into Equation E.23:

$$p_{peri^+,est} = p_{top^+} + \rho_0 g H \int_0^{1/y} \exp\left(-r' + xy^{1/2} r'^{1/2}\right) dr' \tag{E.24}$$

Let:

$$q^2 = r' \tag{E.25}$$

Substituting q into Equation E.24:

$$p_{peri^+,est} = p_{top^+} + \rho_0 g H \int_0^{1/\sqrt{y}} \exp\left(-q^2 + xy^{1/2} q\right) 2q dq \tag{E.26}$$

Rearranging by completing the square:

$$p_{peri^+,est} = p_{top^+} + 2\rho_0 g H \int_0^{1/\sqrt{y}} \exp\left(-\left(q - \frac{xy^{1/2}}{2}\right)^2 + \frac{x^2 y}{4}\right) q dq \tag{E.27}$$

Moving the constant term outside the integration:

$$p_{peri^+,est} = p_{top^+} + 2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right) \int_0^{1/\sqrt{y}} \exp\left(-\left(q - \frac{xy^{1/2}}{2}\right)^2\right) q dq \tag{E.28}$$

Let:

$$s = q - \frac{xy^{1/2}}{2} \tag{E.29}$$

Substituting s into Equation E.28:

$$p_{peri^+,est} = p_{top^+} + 2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right) \int_{0-\frac{xy^{1/2}}{2}}^{\frac{1}{\sqrt{y}}-\frac{xy^{1/2}}{2}} \exp\left(-s^2\right)\left(s + \frac{xy^{1/2}}{2}\right) ds \quad \text{(E.30)}$$

Separating the two summed terms comprising the integrand:

$$p_{peri^+,est} = p_{top^+} + 2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right) \int_{0-\frac{xy^{1/2}}{2}}^{\frac{1}{\sqrt{y}}-\frac{xy^{1/2}}{2}} s \exp\left(-s^2\right) ds + \quad \text{(E.31)}$$
$$2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right) \frac{xy^{1/2}}{2} \int_{0-\frac{xy^{1/2}}{2}}^{\frac{1}{\sqrt{y}}-\frac{xy^{1/2}}{2}} \exp\left(-s^2\right) ds$$

Performing the two integrations:

$$p_{peri^+,est} = p_{top^+} + \frac{2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right)}{-2} \left[\exp\left(-s^2\right)\right]_{0-\frac{xy^{1/2}}{2}}^{\frac{1}{\sqrt{y}}-\frac{xy^{1/2}}{2}} + \quad \text{(E.32)}$$
$$2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right) \frac{xy^{1/2}}{2} \frac{\sqrt{\pi}}{2} \left[erf\left(s\right)\right]_{0-\frac{xy^{1/2}}{2}}^{\frac{1}{\sqrt{y}}-\frac{xy^{1/2}}{2}}$$

Where I have used:

$$\frac{2}{\sqrt{\pi}} \int_0^s \exp\left(-f^2\right) df = erf\left(s\right) \quad \text{(E.33)}$$

Substituting Equations E.19 and E.20 into Equation E.17 and then substituting that into Equation E.32:

$$p_{peri^+,est} = \rho_0 g H \frac{\exp\left(\frac{-1}{y}\right) \exp\left(x\right)}{1 - x\left(1 - \left(1 - y\right)^{1/2}\right)} - \quad \text{(E.34)}$$

$$\rho_0 g H \exp\left(\frac{x^2 y}{4}\right)\left(\exp\left(-\frac{1}{y}+x-\frac{x^2 y}{4}\right)-\exp\left(-\frac{x^2 y}{4}\right)\right)+$$
$$2\rho_0 g H \exp\left(\frac{x^2 y}{4}\right)\frac{x y^{1/2}}{2}\frac{\sqrt{\pi}}{2}\left(erf\left(\frac{1}{\sqrt{y}}-\frac{x y^{1/2}}{2}\right)-erf\left(-\frac{x y^{1/2}}{2}\right)\right)$$

Cancelling terms:

$$\frac{p_{peri+,est}}{\rho_0 g H}=\frac{\exp\left(\frac{-1}{y}\right)\exp(x)}{1-x\left(1-(1-y)^{1/2}\right)}-\left(\exp\left(-\frac{1}{y}+x\right)-1\right)+ \tag{E.35}$$
$$\exp\left(\frac{x^2 y}{4}\right)\frac{x y^{1/2}}{2}\sqrt{\pi}\left(erf\left(\frac{1}{\sqrt{y}}-\frac{x y^{1/2}}{2}\right)-erf\left(-\frac{x y^{1/2}}{2}\right)\right)$$

Combining similar terms:

$$\frac{p_{peri+,est}}{\rho_0 g H}=1+\exp\left(\frac{-1}{y}\right)\exp(x)\left(\frac{x\left(1-(1-y)^{1/2}\right)}{1-x\left(1-(1-y)^{1/2}\right)}\right)+ \tag{E.36}$$
$$\exp\left(\frac{x^2 y}{4}\right)\frac{x y^{1/2}}{2}\sqrt{\pi}\left(erf\left(\frac{1}{\sqrt{y}}-\frac{x y^{1/2}}{2}\right)-erf\left(-\frac{x y^{1/2}}{2}\right)\right)$$

Periapsis pressure can be estimated from this equation. By symmetry, the other leg's value of periapsis pressure is the same except for replacing x with $-x$. Using Equation E.35 and its twin for $p_{peri-,est}$, an expression can be found for:

$$E=2\frac{p_{peri+,est}-p_{peri-,est}}{p_{peri+,est}+p_{peri-,est}} \tag{E.37}$$

E represents how measurable the effects of winds are on a density profile. In the limit that the winds are zero $v_\phi=0$, $x=0$, and so $E=0$. I now simplify by assuming:

$$x \ll 1 \tag{E.38}$$

$$y \ll 1 \tag{E.39}$$

Which leads to:

$$\exp\left(\frac{-1}{y}\right) \rightarrow \exp\left(\frac{-1}{y}\right) \tag{E.40}$$

$$\exp(x) \rightarrow 1 + x \tag{E.41}$$

$$(1-y)^{1/2} \rightarrow 1 - \frac{y}{2} \tag{E.42}$$

$$\exp\left(\frac{x^2 y}{4}\right) \rightarrow 1 + \frac{x^2 y}{4} \tag{E.43}$$

Since:

$$erf(s+\delta s) = erf(s) + \delta s \frac{\partial erf(s)}{\partial s} \tag{E.44}$$

And:

$$\frac{\partial erf(s)}{\partial s} = \frac{2}{\sqrt{\pi}} \exp\left(-s^2\right) \tag{E.45}$$

$$erf\left(\frac{1}{\sqrt{y}} - \frac{xy^{1/2}}{2}\right) \rightarrow erf\left(\frac{1}{\sqrt{y}}\right) - \frac{xy^{1/2}}{\sqrt{\pi}} \exp\left(\frac{-1}{y}\right) \tag{E.46}$$

$$erf\left(\frac{-xy^{1/2}}{2}\right) \rightarrow \frac{-xy^{1/2}}{2} \tag{E.47}$$

Putting all this into Equation E.36:

$$\frac{p_{peri^+,est}}{\rho_0 g H} = 1 + \exp\left(\frac{-1}{y}\right)(1+x)\left(\frac{x\,(1-(1-y/2))}{1-x\,(1-(1-y/2))}\right) + \left(1+\frac{x^2 y}{4}\right)\frac{xy^{1/2}}{2}\sqrt{\pi}\left(erf\left(\frac{1}{\sqrt{y}}\right) - \frac{xy^{1/2}}{\sqrt{\pi}}\exp\left(\frac{-1}{y}\right) + \frac{xy^{1/2}}{2}\right) \tag{E.48}$$

The messy fraction simplifies to $xy/2$:

$$\frac{p_{peri^+,est}}{\rho_0 g H} = 1 + \exp\left(\frac{-1}{y}\right)(1+x)\frac{xy}{2} + \frac{xy^{1/2}}{2}\sqrt{\pi}\left(erf\left(\frac{1}{\sqrt{y}}\right) + \frac{xy^{1/2}}{2}\right) \tag{E.49}$$

Where I have used $\exp(-1/y) \ll 1$ to eliminate part of the last term on the right hand side. Now, since $erf\left(1/\sqrt{y}\right) \rightarrow 1$:

$$\frac{p_{peri^+,est}}{\rho_0 g H} = 1 + \exp\left(\frac{-1}{y}\right)\frac{xy}{2} + \frac{xy^{1/2}}{2}\sqrt{\pi} \tag{E.50}$$

Since $\exp(-1/y) \ll 1$:

$$\frac{p_{peri^+,est}}{\rho_0 g H} = 1 + \frac{xy^{1/2}}{2}\sqrt{\pi} \tag{E.51}$$

$p_{peri^-,est}$ is given by the same equation with the sign of x reversed. Errors in p_{top^+} arising from my crude estimate of the scale height do not influence this result. Using Equation E.37, E is:

$$E = xy^{1/2}\sqrt{\pi} = \frac{\sqrt{2}\,(\Delta r)^{1/2}\,v_\phi^2}{r_0^{1/2} gHt}\left(\frac{H}{\Delta r}\right)^{1/2}\sqrt{\pi} \tag{E.52}$$

Rearranging:

$$E = \left(\frac{2\pi}{r_0 H}\right)^{1/2}\frac{v_\phi^2}{gt} \tag{E.53}$$

For this quasi-cyclostrophic case, the latitudinal pressure gradient is given by:

$$\frac{v_\phi^2}{gH\tan\theta} = \frac{\partial \ln p}{\partial \theta} \tag{E.54}$$

For the quasi-geostrophic case, the latitudinal pressure gradient is given by:

$$\frac{2\Omega v_\phi r\cos\theta}{gH} = \frac{\partial \ln p}{\partial \theta} \tag{E.55}$$

Similarly to before, I assume that $\cos\theta$ is constant and label it c. Since $\Delta r \ll r_0$, I can approximate r in Equation E.55 as r_0 and the above derivation remains valid as long as the following transformation is made:

$$\frac{v_\phi^2}{t} \rightarrow 2\Omega v_\phi r_0 c \tag{E.56}$$

This time x and y are given by:

$$x = \frac{\sqrt{2}\,(\Delta r)^{1/2}\,2\Omega v_\phi r_0 c}{r_0^{1/2} gH} \tag{E.57}$$

$$y = \frac{H}{\Delta r} \tag{E.58}$$

$$E = xy^{1/2}\sqrt{\pi} = \left(\frac{2\pi r_0}{H}\right)^{1/2} \frac{2\Omega v_\phi c}{g} \tag{E.59}$$

For Titan, I assume that $\Delta r = 250$ km and $t = 1$ and take all the other parameters from Table 3.1. I find that $E \sim 0.05$.

For Mars, I assume that $\Delta r = 30$ km and $c = 0.7$ and take all the other parameters from Table 3.1. I find that $E \sim 0.2$.