APPENDIX D

SIMPLIFYING THE VISCOUS TERMS IN EQUATIONS 3.6 – 3.7

In Equation 3.1 the viscous term is $\underline{\nabla} \times (\eta \underline{\nabla} \times \underline{v})/\rho$. This is too complicated to include directly in the scale analysis of Section 3.2.2. Since η is molecular, not eddy, viscosity, I assume that it is uniform and bring it outside the spatial derivatives. I need to find a reasonable approximation for $\underline{\nabla} \times (\underline{\nabla} \times \underline{v})$ in spherical polar coordinates:

$$\begin{aligned}\underline{\nabla} \times \underline{v} = & \frac{1}{r^2 \sin\theta} \underline{\hat{r}} \left(\frac{\partial}{\partial \theta} (r v_\phi \sin\theta) - \frac{\partial}{\partial \phi} (r v_\theta) \right) - \\ & \frac{1}{r \sin\theta} \underline{\hat{\theta}} \left(\frac{\partial}{\partial r} (r v_\phi \sin\theta) - \frac{\partial}{\partial \phi} (v_r) \right) + \\ & \frac{1}{r} \underline{\hat{\phi}} \left(\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial}{\partial \theta} (v_r) \right)\end{aligned} \tag{D.1}$$

The importance of the various terms can be estimated by using Equations 3.14 – 3.16 again:

$$\frac{\partial}{\partial r} \sim \frac{1}{H} \tag{D.2}$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} \sim \frac{1}{R} \tag{D.3}$$

$$\frac{1}{r} \frac{\partial}{\partial \phi} \sim \frac{1}{R} \tag{D.4}$$

Hence:

$$\underline{\nabla} \times \underline{v} = \frac{1}{R}\hat{\underline{r}}\left(\frac{Rv_\phi}{R} - \frac{Rv_\theta}{R}\right) - \frac{1}{R}\hat{\underline{\theta}}\left(\frac{Rv_\phi}{H} - v_r\right) + \frac{1}{R}\hat{\underline{\phi}}\left(\frac{Rv_\theta}{H} - v_r\right) \tag{D.5}$$

Since $H \ll R$:

$$\underline{\nabla} \times \underline{v} = -\hat{\underline{\theta}}\frac{v_\phi}{H} + \hat{\underline{\phi}}\frac{v_\theta}{H} \tag{D.6}$$

I use this approximation for $\underline{\nabla} \times \underline{v}$ to find the dominant terms in $\underline{\nabla} \times (\underline{\nabla} \times \underline{v})$:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \frac{1}{r^2 \sin\theta}\hat{\underline{r}}\left(\frac{\partial}{\partial\theta}\left(\frac{rv_\theta \sin\theta}{H}\right) - \frac{\partial}{\partial\phi}\left(\frac{-rv_\phi}{H}\right)\right) - \frac{1}{r\sin\theta}\hat{\underline{\theta}}\left(\frac{\partial}{\partial r}\left(\frac{rv_\theta \sin\theta}{H}\right) - 0\right) + \frac{1}{r}\hat{\underline{\phi}}\left(\frac{\partial}{\partial r}\left(\frac{-rv_\phi}{H} - 0\right)\right) \tag{D.7}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \frac{1}{R}\hat{\underline{r}}\left(\frac{Rv_\theta}{HR} + \frac{Rv_\phi}{RH}\right) - \frac{1}{R}\hat{\underline{\theta}}\left(\frac{Rv_\theta}{HH}\right) + \frac{1}{R}\hat{\underline{\phi}}\left(\frac{-Rv_\phi}{HH}\right) \tag{D.8}$$

The dominant terms are:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = -\hat{\underline{\theta}}\frac{v_\theta}{HH} - \hat{\underline{\phi}}\frac{v_\phi}{HH} \tag{D.9}$$

Reintroducing the derivatives in their proper form, I have the following approximation for $\underline{\nabla} \times (\underline{\nabla} \times \underline{v})$:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = -\underline{\hat{\theta}} \frac{\partial^2 v_\theta}{\partial r^2} - \underline{\hat{\phi}} \frac{\partial^2 v_\phi}{\partial r^2} \qquad \text{(D.10)}$$