APPENDIX C

CURVATURE TERMS IN EQUATIONS 3.6 – 3.7

Most of the terms in Equation 3.1 have an obvious, one-to-one correspondence with terms in Equations 3.6 – 3.8. The exceptions are the non-linear advective terms.

$$\begin{aligned}(\underline{v}.\underline{\nabla})\,\underline{v} = \underline{\hat{r}}\,(\underline{v}.\underline{\nabla})\,v_r + v_r\,(\underline{v}.\underline{\nabla})\,\underline{\hat{r}} + \underline{\hat{\theta}}\,(\underline{v}.\underline{\nabla})\,v_\theta +\\ v_\theta\,(\underline{v}.\underline{\nabla})\,\underline{\hat{\theta}} + \underline{\hat{\phi}}\,(\underline{v}.\underline{\nabla})\,v_\phi + v_\phi\,(\underline{v}.\underline{\nabla})\,\underline{\hat{\phi}}\end{aligned} \tag{C.1}$$

Equation 3.9 defines $(\underline{v}.\underline{\nabla})$. Using Arfken and Weber (1995), I have:

$$\frac{\partial\underline{\hat{r}}}{\partial r} = 0 \tag{C.2}$$

$$\frac{\partial\underline{\hat{r}}}{\partial\theta} = \underline{\hat{\theta}} \tag{C.3}$$

$$\frac{\partial\underline{\hat{r}}}{\partial\phi} = \sin\theta\underline{\hat{\phi}} \tag{C.4}$$

$$\frac{\partial\underline{\hat{\theta}}}{\partial r} = 0 \tag{C.5}$$

$$\frac{\partial \underline{\hat{\theta}}}{\partial \theta} = -\underline{\hat{r}} \tag{C.6}$$

$$\frac{\partial \underline{\hat{\theta}}}{\partial \phi} = \cos\theta \underline{\hat{\phi}} \tag{C.7}$$

$$\frac{\partial \underline{\hat{\phi}}}{\partial r} = 0 \tag{C.8}$$

$$\frac{\partial \underline{\hat{\phi}}}{\partial \theta} = 0 \tag{C.9}$$

$$\frac{\partial \underline{\hat{\phi}}}{\partial \phi} = -\underline{\hat{r}} \sin\theta - \cos\theta \underline{\hat{\theta}} \tag{C.10}$$

Using Equations C.2 – C.10 to differentiate the unit vectors, the advective terms become:

$$\begin{aligned}(\underline{v}.\underline{\nabla})\,\underline{v} = \underline{\hat{r}}\,(\underline{v}.\underline{\nabla})\,v_r + \underline{\hat{\theta}}\,(\underline{v}.\underline{\nabla})\,v_\theta + \underline{\hat{\phi}}\,(\underline{v}.\underline{\nabla})\,v_\phi + \frac{v_r v_\theta \underline{\hat{\theta}}}{r} + \frac{v_r v_\phi \underline{\hat{\phi}}}{r} + \\ \frac{-v_\theta^2 \underline{\hat{r}}}{r} + \frac{v_\theta v_\phi \cos\theta \underline{\hat{\phi}}}{r \sin\theta} + \frac{v_\phi^2}{r\sin\theta}\left(-\underline{\hat{r}}\sin\theta - \cos\theta\underline{\hat{\theta}}\right)\end{aligned} \tag{C.11}$$

Rearranging:

$$\begin{aligned}(\underline{v}.\underline{\nabla})\,\underline{v} = \underline{\hat{r}}\,(\underline{v}.\underline{\nabla})\,v_r + \underline{\hat{\theta}}\,(\underline{v}.\underline{\nabla})\,v_\theta + \underline{\hat{\phi}}\,(\underline{v}.\underline{\nabla})\,v_\phi + \left(\frac{-\left(v_\theta^2 + v_\phi^2\right)}{r}\right)\underline{\hat{r}} + \\ \left(\frac{v_r v_\theta}{r} - \frac{v_\phi^2}{r\tan\theta}\right)\underline{\hat{\theta}} + \left(\frac{v_r v_\theta}{r} + \frac{v_\theta v_\phi}{r\tan\theta}\right)\underline{\hat{\phi}}\end{aligned} \tag{C.12}$$

The terms containing derivatives continue to be called the advective terms, the others are called curvature terms.